Being analytical

Dr Benoît Jones, Tunnelling and Underground Space MSc Course Manager at the University of Warwick, UK, discusses analytical solutions for tunnels

'ANALYTICAL SOLUTIONS', or 'closed-form solutions', are solutions that lend themselves easily to calculation. Mathematicians will cringe at this simplification, but another way of saying this is that if it can be solved in a single equation without iteration, then it is an analytical solution. If you need to iterate and converge on a solution, then it is a 'numerical method'.

The rise and fall of analytical solutions

Analytical solutions have been a useful design tool for tunnel engineers for some 50 years. In that time, computer power has increased enormously, and numerical modelling programs have become ever more user-friendly. Analytical solutions, due to their need to reduce the complexity of soil-structure interactions to a single equation, usually make simplifying assumptions about geometry and about constitutive behaviour. For instance, tunnels are usually circular, and the problem is usually assumed to be either 2D plane strain or axisymmetric, meaning that the longitudinal dimension is ignored. Simple ground and lining constitutive models are used; either elastic, elastoplastic or viscoelastic, and the variation of geotechnical parameters with depth or stratum boundaries are either grossly simplified or ignored. On the other hand, numerical models can model virtually any geometry, construction sequence, variation in geotechnical parameters or constitutive behaviour. The construction sequence can be modelled explicitly in 3D, pore pressures and consolidation can be modelled, and even creep or thermal effects included.

Given the power of numerical modelling and its increasing accessibility, it is surprising that there has been an explosion in the number of papers published on the subject of analytical solutions for tunnels in recent years. A literature search reveals that more than 30 papers have been published with new analytical solutions for tunnels in peer-reviewed journals in the last 5 years. Many more have been published on the same subject without presenting a new method, for example, papers that compare analytical, empirical and numerical results. Google Scholar returns about 14,000 results for scholarly articles if you type in "analytical solution tunnel* NOT physics" (ensure you restrict it to the last 5 years).

In recent years new analytical solutions have appeared that take account of nonlinear anisotropic ground (Vu et al. 2012), the effect of a radially-strengthened cylinder of ground around a tunnel due to the installation of rockbolts (Bobet, 2009; Carranza-Torres, 2009), others that include creep deformation of the ground (Nomikos et al. 2011; Birchall & Osman, 2012), rectangular tunnels under seismic action (Huo et al., 2006), modelling deformation ahead of the lining installation using a gap parameter (Park, 2005), groundwater ingress to tunnels (Kolymbas & Wagner, 2007), modelling segmental linings (Lee et al., 2001; Hinchberger & El Naggar, 2008) and much, much more.

So what is going on?

Given the superiority of numerical modelling, what is the purpose of so much innovation in analytical solutions to allow these simple models to do something just slightly more complicated than they could do before?

The argument that numerical methods are 'computationally expensive' is often put forward in the introductions to papers about analytical solutions, but is this really still the case when a similarly simple 2D numerical model can be created and solved on a standard PC in a few minutes? Even though numerical modelling software can be expensive, this seems like a bit of a red herring. Being a devil's avocado, the real reason for this paradox may be that most of these papers are published by academics, and academics are not always trying to do something useful, sometimes they are just doing something they find interesting. Although odd to us engineers, a lot of scientists do work that has no immediate practical use.

Analytical solutions as part of the design

There may be a better argument to be made in favour of analytical solutions. Duddeck & Erdmann (1985), in a review of structural design models for tunnels in soft soil, wrote:

"In many instances the stress-strain-deformation problem of a tunnelling procedure can be solved only by the application of a numerical analysis, e.g., the finite-element method. The geometry of the opening and the stratigraphic layers of the ground are not simple in most cases. Non-linear material behaviour is involved. The consecutive phases of the process of driving and supporting are important for the final stresses in the linings."

This is a quite strong criticism of analytical solutions. However, they go on to say:

"Comparison of the results on the basis of numerical solutions is, however, very difficult and does not yield easy insight into dependence on soil and design parameters."

Since analytical solutions are easy to use once a spreadsheet has been created and checked, this makes them ideal for parametric studies. For instance, by varying the stiffness of the lining one can immediately see the effect on the bending moments and hoop forces. Therefore, analytical solutions are useful in 'feeling your way' around a problem and determining which factors are important. The simplicity of the model helps by limiting the complexity of interaction between different factors that may make it difficult to see what is going on.

In addition, many numerical modelling experts use analytical solutions to validate their models. If a numerical model is made to mimic the same boundary conditions and constitutive behaviour as an exact analytical solution, then it should yield the same results. If not, it indicates that there is a problem with the mesh, the element type, the boundary conditions, the convergence criteria or the solution algorithm. Most commercially-available programs include simple examples, which can be compared to analytical solutions. The first steps in any campaign of numerical modelling should be to build simple models and validate them.

Once the simple case has been validated, layers of complexity can be added to a numerical model, one at a time, ensuring the input is debugged and the results are checked at each stage. This is also a kind of parametric study, and enhances understanding of the problem, but also allows easier debugging because if strange results are obtained you know that it is likely to be the newest layer of complexity that caused them. Many of us have learned through bitter experience that to try to jump immediately to the most sophisticated model makes it almost impossible to find out what is going wrong, and makes it difficult to know what behaviour should be expected.

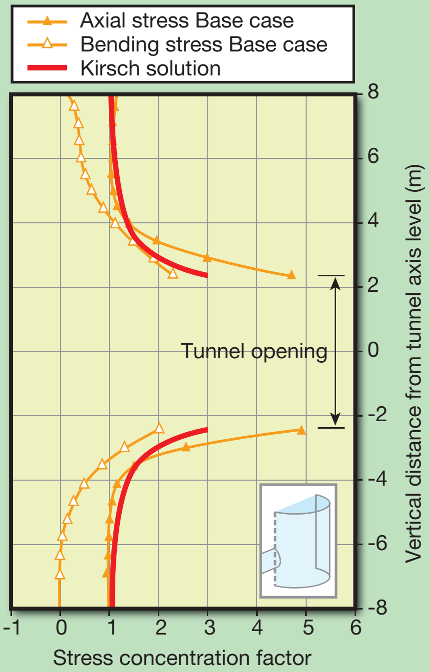
So analytical solutions help us feel our way through those initial stages of tunnel design, and give us more confidence, just as empirical data, if available, help us to refine numerical models and have more confidence in the results as the sophistication of the model increases.

An example

The problem of stress concentrations at tunnel junctions is highly complex, depending not just on the geometry, but being strongly affected by soil-structure interaction and the construction sequence. A major issue in numerical modelling of junctions is to ensure that the element type and mesh refinement of the shell elements representing the lining will accurately predict the large stress gradients around the opening. However, there are no analytical solutions for a tunnel junction situation. The best we can do is an analytical solution for a hole in a plane stress plate developed by Kirsch in 1898 (see Hoek & Brown, 1980). This solution does not calculate the significant bending moments that will be present in the lining due to curvature of the lining, construction sequence and soil-structure interaction, or the effect of these on the axial forces, but it can still be useful.

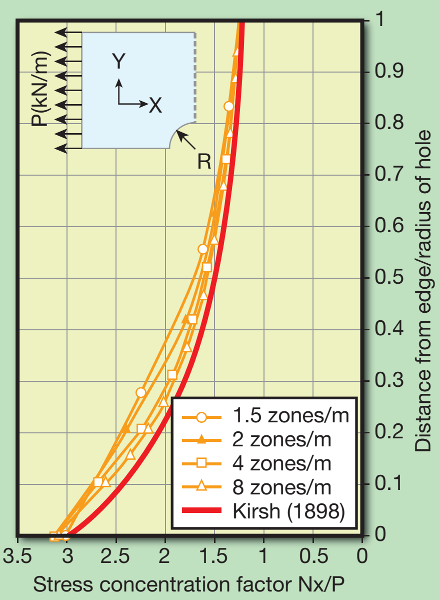
In this case, a shaft-tunnel junction was being modelled (Jones, 2007). A mesh similar to that

Figure 2: Comparison of stress concentration factor for hoop stress in the shaft lining above and below the tunnel centreline with Kirsch analytical solution.



to be used for the shaft lining, but flattened (or 'unwrapped'), was used to mimic the conditions in the analytical solution. The mesh refinement was then varied and the ability of the model to replicate the analytical solution can be seen in Figure 1 for the axial hoop stress above and below the opening. Similar graphs were plotted for hoop stress and vertical stress in the shaft lining along a horizontal line at tunnel axis level. This exercise can be used to inform the mesh geometry used in the numerical model. For a different example, see Pound (2006). Figure 2 shows a comparison between the Kirsch analytical solution and the numerical results from the 3D modelling, again for the hoop stress above and below the shaft opening. A comparison like this allows us to see what the effects of the additional complexity are (curvature effects of the 3D geometry, soil structure interaction and construction sequence, at the stage of breaking out the opening for the tunnel from the shaft), compared to the very simple case of a plane stress plate with a hole in it. This enhances our understanding of the problem. The axial hoop stresses in the 3D numerical model are higher than for the plane stress plate, and this is due to the three-dimensional deformation of the shaft and the soil-structure interaction. Essentially, when the opening is made, the ground arches onto the shaft lining around the opening, increasing the radial stress and hence the axial stresses. In addition, the opening wants to close at axis and wants to increase in diameter vertically and push into the ground above and below the opening. None of these effects can be modelled in the analytical solution, but the comparison helps us to separate them and understand them. Also on Figure 2, the bending stresses from the 3D numerical model are plotted, and this demonstrates why the analytical solution is of limited use to a designer – the analytical model is incapable of predicting these stresses, which are almost as large as the axial stresses and are therefore very significant. But the validation of the model would not have been possible without the analytical solution and it is an important step in the process.

Figure 1: Stress concentration factor for a plane stress plate with a hole in the direction parallel to the applied stress (Nx/P) along a line transverse to the direction of the applied stress.



Conclusions

Analytical solutions, although largely supplanted by numerical modelling for final design, still have a role to play. They can help us to 'feel our way' at an early stage of design, simplifying the problem and giving insight into the influence of different parameters. They help us validate numerical models as part of an incremental design process of increasing complexity. In short, they are still of value and are here to stay.

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