

## **Guidelines for Gaussian curve-fitting to settlement data**

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**ABSTRACT:** Fitting a Gaussian curve to real settlement data is of practical use because the parameters that describe the curve, namely volume loss, maximum settlement and trough width, are conceptually easy to understand and can be compared at different locations and for different tunnel sizes, depths and construction methods. Where the magnitude of surface settlements is small, it can be difficult to obtain the Gaussian curve parameters because the magnitude of the measurement error is approaching the magnitude of the surface settlements themselves. Other factors influencing the accuracy of curve-fitting are the number of monitoring points and their spacing relative to the trough width, and the curve-fitting method used. This paper presents the results of a Monte Carlo analysis, which are then used to provide guidance on the reliability of Gaussian curve parameter determination for different error ratios, trough widths, transverse settlement monitoring array layouts and curve-fitting methods.

### **1 Introduction**

As the underground space in our urban centres becomes increasingly congested, linear tunnels are being constructed ever deeper to avoid existing underground infrastructure. Although deeper tunnels generally present less risk of excessive settlement at the surface, it is still necessary to attempt to predict those settlements, and to predict the larger settlements that may be experienced by underground structures much closer to the tunnel horizon.

Predictions of ground movements due to tunnelling rely heavily on empirical methods, which are based on published case studies of tunnels constructed in similar ground conditions. The pattern of surface settlement induced transverse to a tunnel under construction may be characterised by a 'Gaussian' settlement trough, attributed to Peck (1969) and Schmidt (1969), which takes the form of an inverted normal (or Gaussian) distribution curve transverse to the direction of the tunnel drive. This is illustrated in Figure 1.

For relatively low-risk tunnels, where predicted settlements are small, the level of risk will not usually merit the use of sophisticated surveying methods and the disruption at the surface they may cause. In these cases, the industry best practice method of monitoring surface settlements is to use a precise laser level and bar-coded invar staff to level a transverse array of monitoring points, consisting of road nails inserted into the pavement or road surface. Where possible, surfacing types such as paving slabs that may be unstable are avoided in favour of more solid objects, such as kerb stones. Although some monitoring points do experience instability, if sufficient background readings are taken these points can be easily identified and removed from the dataset. The repeatability of this surveying method, estimated from a large number of background readings, has been found to be better than  $\pm 0.5\text{mm}$  (Jones, 2010).

Where this surface settlement monitoring method is used, and the magnitude of surface settlements is small (for example, a centreline settlement of 1 to 3mm), it can be difficult to obtain the Gaussian curve parameters because the magnitude of the potential measurement error is of the same order as the magnitude of the surface settlements themselves. Guidance is required on the best objective method of curve-fitting, and what confidence one can have in the derived Gaussian curve parameters for a given arrangement of monitoring points and a given magnitude of settlement.

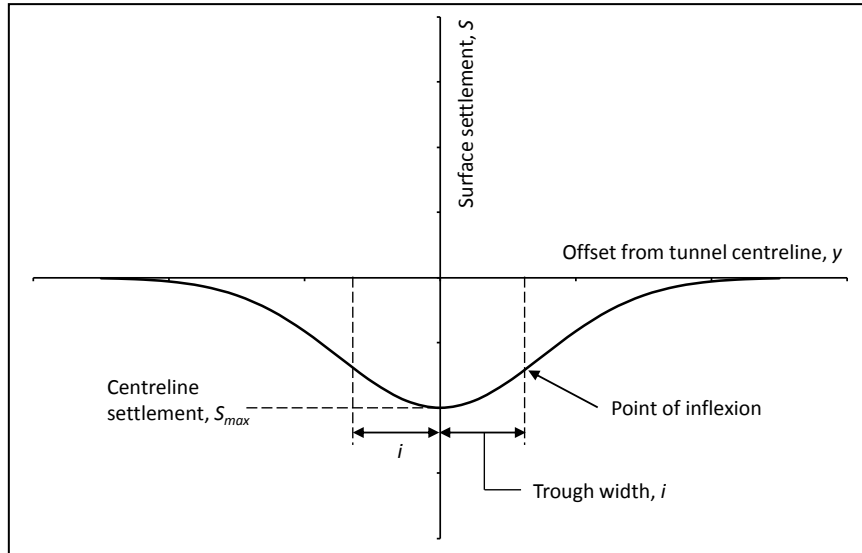


Figure 1. Gaussian settlement trough

## 2 Gaussian curve-fitting methods

In this section, the equations that define a Gaussian curve will be described. Then various methods of fitting a Gaussian curve to surface settlement data will be explained.

### 2.1 The Gaussian curve

The distance from the centreline to the point of inflexion of the curve, which in a normal distribution would be the standard deviation, is known as the 'trough width',  $i$ . The maximum settlement over the centreline of the tunnel  $S_{max}$  is analogous to the mean of a normal distribution. Settlement  $S$  at offset  $y$  from the tunnel centreline is therefore given by:

$$S = S_{max} \exp\left(-y^2/2i^2\right) \quad (1)$$

The area under the curve is defined as the 'volume loss',  $V_s$ . This may be given by the following equation:

$$V_s = \int_{-\infty}^{\infty} S dy = \int_{-\infty}^{\infty} S_{max} \exp\left(-y^2/2i^2\right) dy = \sqrt{2\pi} i S_{max} \quad (2)$$

Using Equations 1 and 2 or combinations thereof, the curve may be defined by any two of the parameters  $V_s$ ,  $S_{max}$  or  $i$ . Therefore, curve-fitting is not straightforward as there are two variables. The actual methods used to fit a Gaussian curve to real data are seldom described in the literature. Only New & Bowers (1994) mention the use of nonlinear regression analysis to calculate Gaussian curve parameters. Nowhere, except in Jones (2010), are the methods used actually described in any detail.

### 2.2 Nonlinear regression

When there are two variables, nonlinear regression may be adopted to fit a curve. This would involve varying both  $V_s$  and  $i$  to find the 'best fit'. In this case the 'best fit' was defined as the minimum sum of absolute errors (SAE). Here 'error' is defined as the difference between the measured monitoring data value and the Gaussian curve value at the same offset from the tunnel centerline  $y$ . These errors are also known as 'residuals' in regression analysis.

When implemented in a spreadsheet, this method creates a large table with volume loss incrementally varying in each row, and trough width incrementally varying in each column. These values can be refined until a sufficiently precise solution is bracketed. A simplified example is shown in Table 1.

**Table 1. Simplified example table for nonlinear regression method**

Trough volume (mm.m)	Trough width (m)				
	10.0	10.5	11.0	11.5	etc...
0.3					
0.4					
0.5					
etc...					

In each cell of Table 1, the following equation was used to calculate SAE:

$$SAE = \sum_{j=1}^n \left| \frac{V_s}{\sqrt{2\pi} \cdot i} \exp\left(\frac{-y_j^2}{2i^2}\right) - S_j \right| \quad (3)$$

Where  $V_s$  and  $i$  vary from cell to cell, and  $S_j$  is data point  $j$  from a settlement monitoring array with  $n$  monitoring points. In each cell, the sum of the absolute errors (SAE) is calculated for the corresponding values of trough volume and trough width. The minimum value of SAE is then found by searching the table and the corresponding values of volume loss and trough width describe the best fit Gaussian curve.

### 2.3 Direct calculation methods

A simpler method to nonlinear regression would be to use trapezoidal integration to estimate trough volume or volume loss, and then to calculate the trough width in one of two ways.

The measured volume loss using trapezoidal integration,  $V_{sm}$ , is given by:

$$V_{sm} = \frac{\sum_{j=1}^{n-1} \left[ \left( \frac{S_j + S_{j+1}}{2} \right) (y_j - y_{j+1}) \right]}{A_f} \quad (4)$$

where  $j$  is the point number from 1 to  $n$ ,

$n$  is the number of points in an array,

$y_j$  is the transverse distance of point  $j$  from the tunnel centreline,

$S_j$  is the measured settlement of point  $j$ ,

and  $A_f$  is the excavated face area of the tunnel.

The numerator of the quotient in Equation 4 is basically a trapezoidal integration of the settlement data. If the data do not cover the whole of the settlement trough, for example because the trough width was larger than expected or because points could not be installed due to the presence of buildings or other obstructions, the actual volume loss,  $V_s$ , may be estimated by using the following equation:

$$V_s = \frac{V_{sm}}{\int_{-\infty}^{x_n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y^2}{2\sigma^2}\right)} - \int_{-\infty}^{x_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y^2}{2\sigma^2}\right)} dy} \quad (5)$$

where the two terms in the denominator represent the unit cumulative distribution from minus infinity to  $x_1$  and  $x_n$ , i.e. the denominator is the proportion of the total volume loss that is within the limits of the array.

Trapezoidal integration is reasonably accurate, since the missing volumes due to curvature between data points in the hogging and sagging parts of the curve to an extent cancel each other out. The accuracy depends on the spacing of the monitoring points and their locations relative to the trough width.

The first method of trough width calculation (called 'DCJ' in the analysis to follow) exploits the analogy of trough width  $i$  to the error function standard deviation. When a Gaussian curve is used to represent ground movements due to tunnelling, by analogy the standard deviation or the point of inflexion is the trough width,  $i$ , and the frequencies are the settlements,  $S$ . Therefore, the trough width may be

calculated directly from the data by calculating the standard deviation about a mean assumed to be at the centreline of the tunnel.

The standard deviation  $\sigma$ , or the trough width  $i$ , is given by:

$$\sigma = \sqrt{\frac{\sum_{j=1}^n [y_j^2 S_j]}{\sum_{j=1}^n [S_j]}} = i \quad (6)$$

where  $S_j$  is the measured settlement of point  $j$ .

This equation is sensitive to errors at large offsets, so judgment should be exercised to exclude errors at large offsets where the settlement should be negligible. The equation may also be less accurate when the spacing of monitoring points is not constant, since it is a standard equation for a histogram of frequencies.

The second method (called 'DCSMAX' in the analysis to follow) uses a trapezoidal integration to calculate volume loss, in the exactly the same manner as for the DCJ method. The trough width,  $i$ , is calculated for each value of settlement  $S_j$  by reformulating Equation 1 as follows:

$$i = \sqrt{\frac{-y^2}{2 \cdot \ln(S_j/S_{max})}} \quad (7)$$

The assumption is made that the centreline settlement,  $S_{max}$ , is correct. Since it is usually the largest settlement in an array, measurement error is likely to have less effect, proportionally. Trough width,  $i$ , is calculated for each point in the array (except the centreline point), and the values averaged.

### 3 Monte Carlo analysis

It is not possible to rigorously compare the goodness of fit of Gaussian curve-fitting methods by applying them to real data, since they each use a different criterion for goodness of fit. However, a comparison may be made by using each of these methods in a Monte Carlo analysis with a large number of fictitious monitoring data sets.

Each fictitious data set begins with the same 'perfectly Gaussian' set of data at fixed offsets from the centreline, with a predetermined volume loss and trough width. Then each data point is assigned a random error. This is a Monte Carlo analysis (rather than a 'What if?' analysis), in that the assigned random errors follow a normal distribution. Therefore, the mean values of volume loss and trough width should approach the original 'perfectly Gaussian' values after a sufficiently large number of randomised data sets have been used. Once convergence has been achieved, the geometric mean and geometric standard deviation of trough width  $i$  and the mean and standard deviation of trough volume  $V_s$  will provide a means of comparing the accuracy of the curve-fitting methods.

#### 3.1 Codes used in the analysis

Codes have been used to simplify the presentation of the analyses. These consist of the array type, the trough width, then the centreline settlement, as shown in the following example:

A7.5-5 = Array type A, trough width 7.5m, centreline settlement 5mm.

#### 3.2 Convergence criteria

A convergence criterion is required to demonstrate when sufficient randomised data sets have been used. At this point, further sampling will not significantly change the values of the mean and standard deviation of trough width and volume loss that have been determined.

For trough width  $i$  this has been defined as when the instantaneous geometric mean and geometric standard deviation do not vary from the overall mean value by more than  $\pm 0.1m$  over the last 50 data sets. The geometric mean and geometric standard deviation were used because the determined values of  $i$  follow a lognormal distribution. This is because  $i$  cannot be less than zero, but in rare cases it can be substantially larger than the expected value, particularly for low error ratios. Lognormal distributions occur when the natural logarithm of a variable follows a normal distribution. This follows

from the form of Equation 1, since the settlements  $S$  follow a normal distribution and  $S$  is related to the exponential of  $i$ .

For volume loss  $V_s$ , convergence has been defined as when the instantaneous mean and standard deviation values of trough volume have not varied from the overall mean value by more than  $\pm 1.0\%$  of the expected mean value of trough volume over the last 50 data sets (e.g. for a theoretical 37.6mm.m trough volume, the mean and standard deviation must not vary by more than  $\pm 0.376\text{mm.m}$ ).

In addition, a minimum number of 100 randomised data sets were used.

### 3.3 Array types

Three types of transverse settlement arrays were used, denoted A, B and C. These are detailed in Table 2 below.

**Table 2. Array types used in Monte Carlo analysis**

Array type	$n$	Offsets of points 1 to $n$ (m)
A	5	-25, -10, 0, 10, 25
B	11	-35, -25, -15, -10, -5, 0, 5, 10, 15, 25, 35
C	21	-50, -45, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50

Array type A represents a settlement monitoring array that could be used for low-risk tunnels, for example a small diameter EPB TBM at depths greater than 20m. Array type B represents a more detailed array with a wider extent, which may be used if better quality information on the Gaussian curve parameters is required. Array type C represents the type of array that has been used in the past for research purposes and has been included to show the effect of using the most detailed array that is reasonably practical using precise levelling. More detail and accuracy could be obtained using a string of electrolevels or tiltsensor beams, for example, but in most cases this is not feasible at the ground surface.

In all three cases, it has been assumed that benchmarks have been installed sufficiently far away from the tunnel's zone of influence and that the settlements measured are absolute values and not relative to the outermost points.

### 3.4 Trough widths

Three trough widths have been used; 7.5m, 12.5m and 20m. This range will cover the vast majority of tunnel depths in a variety of soil types.

### 3.5 Centreline (maximum) settlements

Seven different values of centreline (maximum) settlement  $S_{max}$  have been used; 0.5mm, 1mm, 2mm, 5mm, 10mm, 20mm and 50mm. This range will cover the vast majority of tunnel diameters and tunnel depths with the given range of trough width values.

## 4 Comparison of curve-fitting methods using Monte Carlo analysis

In order for the results of the Monte Carlo analysis to be more easily applicable, the ratio between the centreline settlement  $S_{max}$  and the standard deviation of the random errors applied to the data  $\sigma_{mc}$  has been used. This will be referred to as the 'error ratio'. In general, if  $S_{max}$  is much larger than  $\sigma_{mc}$ , the error ratio will be large and the reliability of curve fitting is expected to be much better than for a small error ratio where the errors are approaching the value of  $S_{max}$ .

$$Error\ ratio = \frac{S_{max}}{\sigma_{mc}} \quad (8)$$

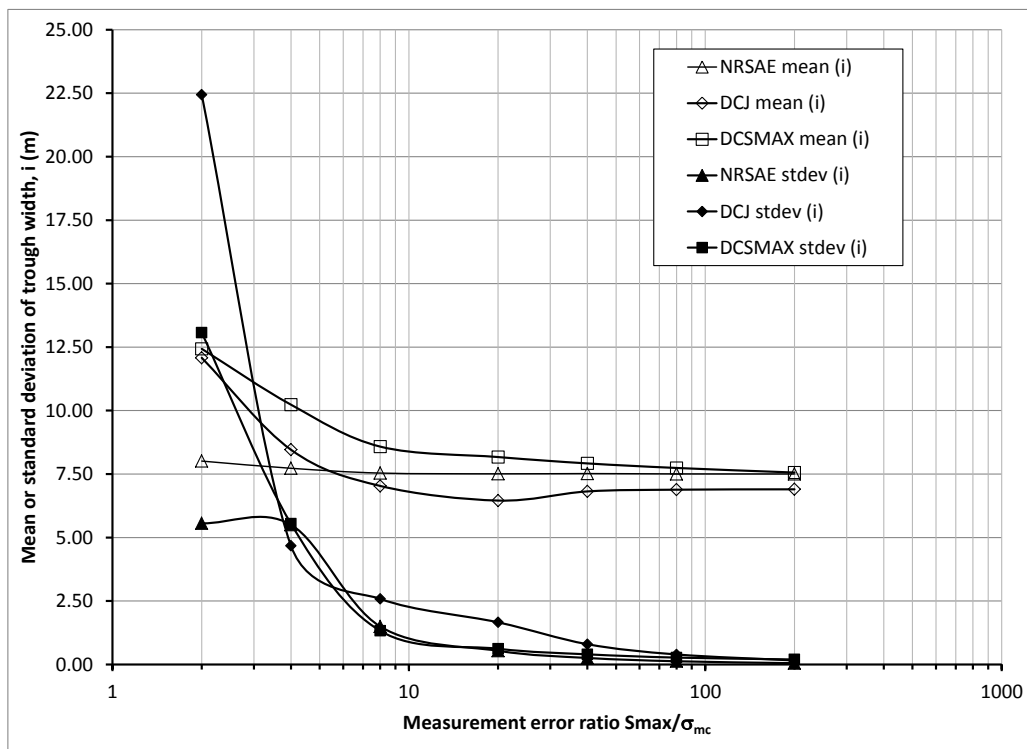
Therefore, Gaussian curve parameters calculated from small values of settlements when simple surveying methods are used for surface settlement monitoring should be less reliable. The aim of the Monte Carlo analyses is to quantify this reliability in terms of confidence limits for trough width and volume loss for a given  $S_{max}$  value and the standard deviation of measurement error.

Since a single value of standard deviation of the applied random errors has been used, equal to 0.25mm, the values of error ratio listed in Table 3 are obtained.

**Table 3. Error ratios used in Monte Carlo analyses**

Centreline settlement, $S_{max}$ (mm)	Error ratio $S_{max}/\sigma_{mc}$
0.5	2
1	4
2	8
5	20
10	40
20	80
50	200

An example of a comparison between the different curve-fitting methods using Monte Carlo analysis is shown in Figure 2. NRSAE is nonlinear regression using the sum of absolute errors method, and DCJ and DCSMAX are the two direct calculation methods. A trend of improved reliability of trough width determination as the error ratio increases can be seen.



**Figure 2. Variation of mean and standard deviation of Monte Carlo trough width values vs. error ratio for array type A, trough width 7.5m**

For the calculation of trough width, at all error ratios the NRSAE method was the most reliable, giving results that converge on the theoretical mean and with generally the lowest value of standard deviation. This basic pattern was repeated for most situations. Although results were different for different array types and trough width values the standard deviation of trough width was generally less than 1.5m for error ratios of 8 and above. For best practice levelling methods this would correspond to a maximum centreline settlement of around 2mm.

Looking at the volume loss, Figure 3 shows the relationship between the percentage error of the Monte Carlo analysis mean with error ratio. Again, a trend of increasing reliability with increasing error ratio may be observed. Interestingly, the direct calculation (DC) of volume loss by trapezoidal integration is more reliable than the nonlinear regression (NRSAE).

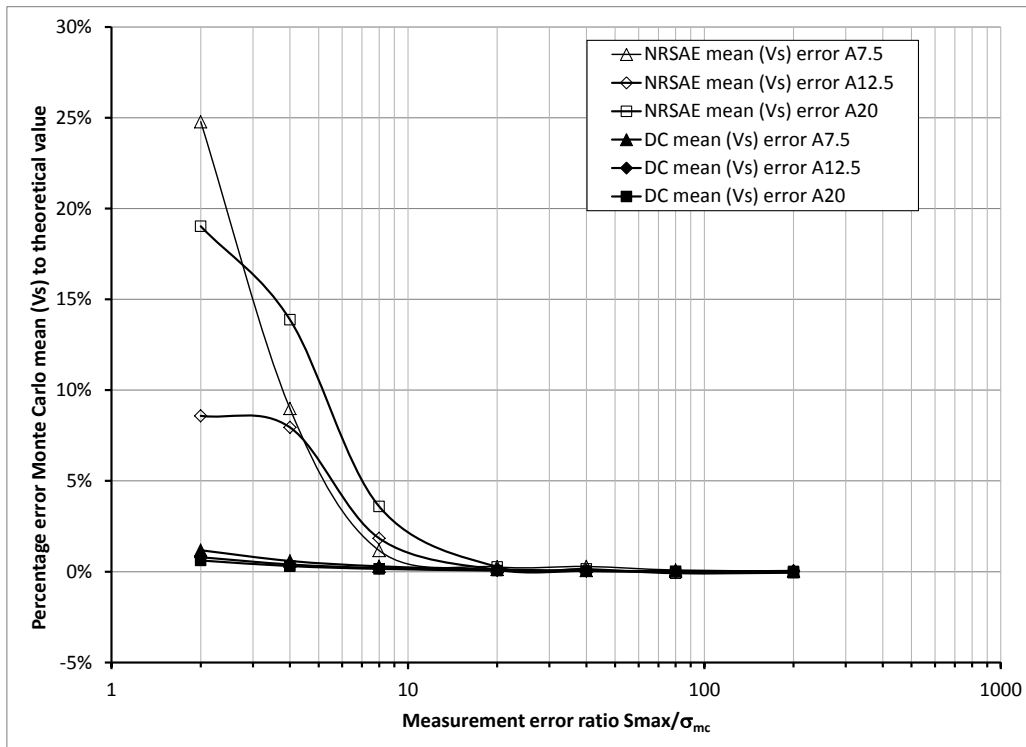


Figure 3. Percentage error of Monte Carlo mean with error ratio for DC and NRSAE methods of trough volume calculation, array type A, trough widths 7.5m, 12.5m and 20m

The real test of reliability is the standard deviation of trough volume. The standard deviation as a percentage of the theoretical trough volume is shown for array type A only in Figure 4. Here it can be seen that for all trough widths the standard deviation is smaller using the direct calculation as opposed to nonlinear regression. Another clear trend is that as the trough width increases the standard deviation of trough volume decreases.

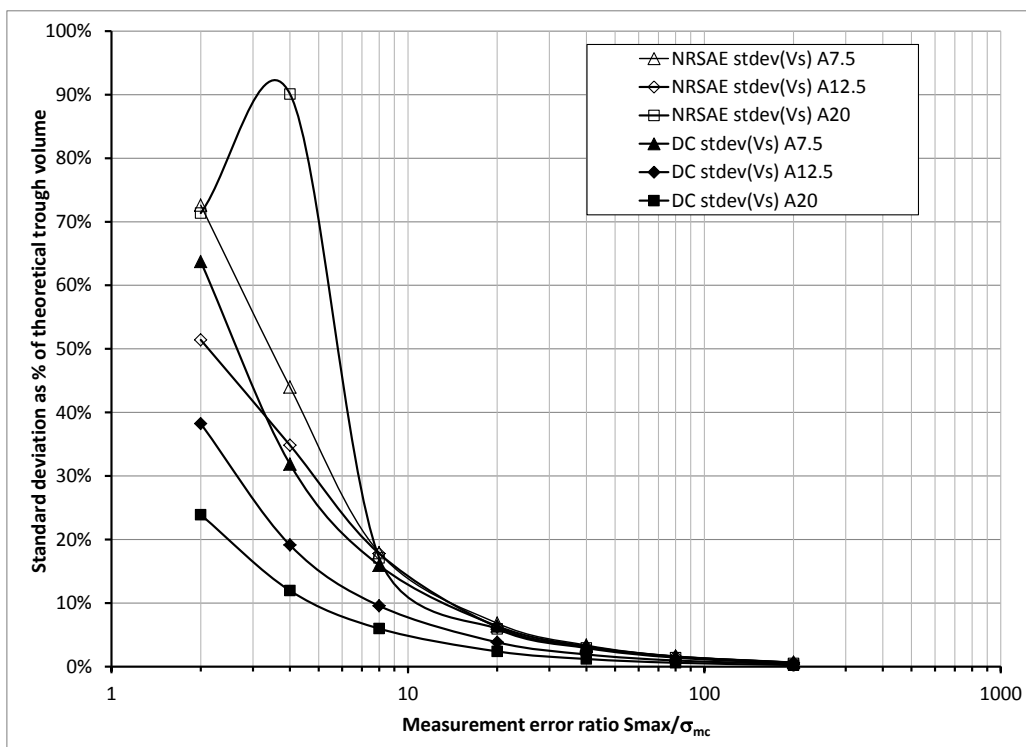


Figure 4. Monte Carlo standard deviation of trough volume as a percentage of theoretical trough volume for array type A, trough widths 7.5m, 12.5m and 20m

There appears to be an anomaly in the NRSAE method at an error ratio of 4 (this is  $S_{max} = 1\text{mm}$ ), which for a trough width of 20m means that the standard deviation is worse than for the error ratio of

2, which is unexpected. This was because it was not always possible to converge the nonlinear regression to a solution where the error ratio was small, as sometimes the errors conspired to make the curve very non-Gaussian. Therefore, for instance, the 'worst' 16 samples were left out of the A20-0.5 calculation of mean and standard deviation, but only 4 of these didn't converge in the A20-1 calculation and so were included. Since the same randomised values were used for the different error ratios, these 12 samples had a significant effect on the A20-1 standard deviation by contributing outliers to the population.

## **5 Conclusions**

Monte Carlo analysis was found to be a valuable means of comparing different Gaussian curve-fitting methods in a rational, repeatable and quantifiable manner.

The results indicate that nonlinear regression is the best method to use for estimating trough width, with estimates of trough width following a lognormal distribution with a geometric mean close to the expected mean and a standard deviation generally lower than the other methods. On the other hand, direct calculation methods were more reliable in the calculation of volume loss, have computational advantages and do not require complex iterative calculations.

A large variety of trough widths and error ratios were analysed, only a selection of which were presented in this paper. The following guidelines have been derived from the larger study:

- Do not use an array similar to type A if a reliable estimate of Gaussian curve parameters is required.
- There is little difference in reliability between array types B and C when the NRSAE method is used.
- For most practical purposes, the lowest error ratio to achieve acceptable estimates of Gaussian curve parameters using array types B or C is approximately 8. This corresponds to a centreline settlement of 2mm used with a monitoring method with a standard deviation of measurement error of 0.25mm.
- As trough width increases, the reliability of trough width estimation from real data will decrease, but the reliability of volume loss estimation will improve.

## **6 Acknowledgements**

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